Integrable $1/r^2$ spin chain with reflecting end

1996 J. Phys. A: Math. Gen. 29 3977

(http://iopscience.iop.org/0305-4470/29/14/021)

J. Phys. A: Math. Gen. **29** (1996) 3977–3984. Printed in the UK

Integrable $1/r^2$ spin chain with reflecting end

Takashi Yamamoto†§ and Osamu Tsuchiya‡‖

† Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606, Japan
‡ Department of Pure and Applied Sciences, University of Tokyo, Komaba, Meguro-ku, Tokyo 153, Japan

Received 26 February 1996

Abstract. A new integrable spin chain of the Haldane–Shastry type is introduced. It is interpreted as the inverse-square interacting spin chain with a *reflecting end*. The lattice sites of this model consist of the square roots of the zeros of the Laguerre polynomial. Using the 'exchange operator formalism', the integrals of motion for the model are explicitly constructed.

Studies of the Calogero–Sutherland model [1], the Haldane–Shastry spin chain [2] and their variants [3] have provided many new links with other areas of physics and mathematics. In particular, these models provide exactly solvable models in which the ideas of fractional exclusion statistics can be tested [4, 5].

In [6], with a view to proving the quantum integrability of the Calogero–Sutherland model and its rational version, the Calogero–Moser model confined in a harmonic potential (which we call the Calogero model), Polychronakos proposed the so-called *exchange operator formalism*. His clever formalism is applicable not only to continuum models but also to spin chain models, and has become a standard technique for the study of the integrability and spectrum of inverse-square interacting systems [7–13]. Within the exchange operator formalism, all of the inverse-square interacting spin chain models can be related to the appropriate continuum inverse-square interacting models with *internal degrees of freedom (spin)*. More precisely, the spin chain models are obtained by freezing out the kinematic degrees of freedom in the corresponding continuum models, and the lattice¶ sites lie at the classical static-equilibrium positions of the continuum models [14–16]. For example [8, 17], the Haldane–Shastry model is related to the spin Calogero–Sutherland model [18, 7, 19] whose classical equilibrium positions form a regular lattice on the circle.

Polychronakos [17] has applied his formalism to constructing the new spin chain model related to the spin Calogero model [7, 10, 11, 20, 21]. We call this model the Polychronakos–Frahm (PF) model [22, 23]. The lattice sites of the PF model are positioned at the zeros of the Hermite polynomial, i.e. the spins are no longer equidistant. Against this unusual property, the spectra of the PF model are equally spaced and therefore simpler than those of the Haldane–Shastry model. Thus the fractional exclusion statistics for the elementary excitations of the PF model are more tractable than those of the Haldane–Shastry model [23].

§ E-mail address: yam@yukawa.kyoto-u.ac.jp
‖ E-mail address: otutiya@hep1.c.u-tokyo.ac.jp
¶ In this paper, we use the term *lattice* in both the usual and an unusual sense. That is, this term does not always imply translational invariance.

0305-4470/96/143977+08$19.50

On the other hand, in [24, 25] another generalization of the spin chain model, the Haldane–Shastry model with open boundary conditions (the BC_N-type Haldane–Shastry model), was introduced. This model is related to the BC_N-type spin Calogero–Sutherland model [26, 27]. It is now well known that such BC_N-type models can be applicable to analysing physics with boundaries [28–31]. In particular, one of the authors and his collaborators have shown that the above models possess the properties of the chiral Tomonaga–Luttinger liquids [31].

The aim of this paper is twofold. The first is to prove the integrability of the B_N-type spin Calogero model [32] within the exchange operator formalism. The second is to construct a new integrable spin chain model related to the B_N-type spin Calogero model. This spin chain model can be thought of as the 'intersection' of the PF model and the BC_N-type Haldane–Shastry model.

Before turning to the explicit calculation, we shall briefly mention this new integrable spin chain. The Hamiltonian is given by

$$\mathcal{H}_{\mathrm{PF}} = \sum_{1\leqslant j\neq k\leqslant N}\left[\frac{1}{(x_j - x_k)^2}P_{jk} + \frac{1}{(x_j + x_k)^2}\overline{P}_{jk}\right] + \gamma\sum_{j=1}^{N}\frac{1}{x_j^2}P_j \tag{1}$$

where N is the number of sites and $\gamma \in \mathbb{R}$ is a parameter. In the above Hamiltonian we have introduced the B_N-type spin exchange operators for the ν-component spin variables [25, 32]; the operator P_{jk} exchanges the spins at the sites j and k, the operator P_j is an involution† for the set of spin variables at the site j, i.e. ${P_j}^2 = 1$, and finally the operator $\overline{P}_{jk}$ is defined by $\overline{P}_{jk} = P_j P_k P_{jk}$. Also it will be shown that, from the integrability condition of the model, the lattice sites x_j lie at the square roots of the zeros of the Laguerre polynomial $L_N^{(|\gamma|-1)}(y)$ (see [33] for the notation). It is well known that the Laguerre polynomial $L_N^{(\alpha)}(y)$ with $\alpha > -1$ $(= -1)$ has N distinct roots, $0 < y_1 < y_2 < \cdots < y_N$ $(0 = y_1 < y_2 < \cdots < y_N)$ [33]. Therefore the lattice of the model is well-defined and does not contain negative sites. Also, it is easy to see that the lattice is not uniform. For example, in the case $N = 4$, $\gamma = 2$, the model has the lattice (0.86, 1.60, 2.39, 3.31).

There are several points which should be noticed in (1). Clearly, the Hamiltonian (1) is not translationally invariant because the lattice is not uniform. Even if we suppose that the lattice is uniform, the terms $\overline{P}_{jk}/(x_j + x_k)^2$ and P_j/x_j^2 in (1) break translational invariance. The term $\overline{P}_{jk}/(x_j + x_k)^2$ represents the interaction between the jth spin and the 'mirror-image' of the kth spin. With an appropriate choice of the representation of the operator P_j, the last term in (1) can be regarded as magnetic fields whose magnitudes are proportional to the inverse-square of the positions of the sites. From these observations, the origin $x = 0$ can be regarded as a reflecting end of the system. Then we call the model with Hamiltonian (1) the PF model with a reflecting end or the B_N-PF model (if $\gamma = 0$, we call it the D_N-PF model).

Consider now the integrability of the B_N-type spin Calogero model. We first recall the B_N-type spin Calogero model. The Hamiltonians of the B_N-type spin Calogero–Moser model and the B_N-type spin Calogero model are respectively given [32] by

$$\bar{H}_{\mathrm{CM}} = \sum_{j=1}^{N}\left[-{\partial_j}^2 + \frac{1}{x_j^2}\beta_1(\beta_1 - M_j)\right] + \sum_{1\leqslant j\neq k\leqslant N}\left[\frac{1}{(x_j - x_k)^2}\beta(\beta - M_{jk}) + \right.$$

† This is a formal definition. If we need a more explicit form of the operator P_j, we must chose an appropriate representation of it. For instance, in the case of $\nu = 2$, the non-trivial representation of the operator P_j can be chosen as the third component of the Pauli matrices σ^3.

$$+\frac{1}{(x_j+x_k)^2}\beta(\beta-\overline{M}_{jk})\Bigg] \qquad (2)$$

$$\bar{H}_{\mathrm{C}}=\bar{H}_{\mathrm{CM}}+\omega^2\sum_{j=1}^{N}x_j^2 \qquad (3)$$

where $\beta, \beta_1 \in \mathbb{R}$ and $\omega \in \mathbb{R}_{\geqslant 0}$ are coupling constants, and $\partial_j = \partial/\partial x_j$. In (2), we have already introduced the operators M_j, M_{jk} and $\overline{M}_{jk}(= M_j M_k M_{jk})$ which are called the B_N-type (coordinate) exchange operators, and are defined by the action on the coordinates x_j:

$$M_{jk}x_j = x_k M_{jk} \qquad M_j x_j = -x_j M_j. \qquad (4)$$

We remark that the two sets of operators $\{M_j, M_{jk}, \overline{M}_{jk}\}$ and $\{P_j, P_{jk}, \overline{P}_{jk}\}$ satisfy the same relations which are the defining relations for the Weyl group of type B_N [32].

The Hamiltonians (2) and (3) do not contain the terms related directly to the spin. The spin degrees of freedom are introduced as follows. Let $\mathbf{\Omega}^s = C^\infty(\mathbb{C}^N)\otimes V$ where V denotes the space of spins, for example, $(\mathbb{C}^\nu)^{\otimes N}$. Then operators M_{jk}, M_j, P_{jk} and P_j naturally act on this space, and clearly M_{jk} and M_j commute with P_{jk} and P_j. Next we introduce a projection π which respectively replaces every occurrence of M_{jk} and M_j by P_{jk} and P_j after M_{jk} and M_j have been moved to the right of the expression. Consider the B_N-type 'bosonic' subspace

$$\widetilde{\mathbf{\Omega}}^s = \{f \in \mathbf{\Omega}^s \mid (M_{jk}-P_{jk})f=0,\ (M_j-P_j)f=0\}. \qquad (5)$$

For any operator $\bar{\mathcal{O}}$, the projection π leads to a unique operator $\mathcal{O}$ which satisfies $\bar{\mathcal{O}}\widetilde{\mathbf{\Omega}}^s = \mathcal{O}\widetilde{\mathbf{\Omega}}^s$ and does not contain the coordinate exchange operators. The Hamiltonians with the spin degrees of freedom are thus given by the operators $\pi(\bar{H}_{\mathrm{CM}})$ and $\pi(\bar{H}_{\mathrm{C}})$. Also, the spinless, i.e. the one-component case, can be considered by putting $P_{jk}=1$, $P_j=1$. In this case, the conditions in (5) are simply the conditions for the B_N-invariance of the wavefunctions.

First of all, we introduce the operators $\mathcal{D}_j$ for later use:

$$\mathcal{D}_j=\sum_{k\neq j}\left[\frac{1}{x_j-x_k}M_{jk}+\frac{1}{x_j+x_k}\overline{M}_{jk}\right]+\frac{\beta_1}{\beta}\frac{1}{x_j}M_j. \qquad (6)$$

It is easy to show that

$$M_j\mathcal{D}_j=-\mathcal{D}_jM_j \qquad M_{jk}\mathcal{D}_j=\mathcal{D}_kM_{jk} \qquad (7)$$

$$[\mathcal{D}_j,\mathcal{D}_k]=0 \qquad (8)$$

$$[\mathcal{D}_j,x_k]=\delta_{jk}\left(-\sum_{l\neq j}(M_{jl}+\overline{M}_{jl})-2\frac{\beta_1}{\beta}M_j\right)+(1-\delta_{jk})(M_{jk}-\overline{M}_{jk}). \qquad (9)$$

Next, as in the ordinary case [34, 35], we define the B_N-type Dunkl operators D_j by

$$D_j=\partial_j-\beta\mathcal{D}_j. \qquad (10)$$

Using $[\partial_j, x_k]=\delta_{jk}$, $M_j\partial_j=-\partial_jM_j$, $M_{jk}\partial_j=\partial_kM_{jk}$, etc, we can show that the B_N-type Dunkl operators D_j together with the coordinates x_j satisfy the following relations:

$$M_jD_j=-D_jM_j \qquad M_{jk}D_j=D_kM_{jk} \qquad (11)$$

$$[D_j,D_k]=0,\ [x_j,x_k]=0 \qquad (12)$$

$$[D_j,x_k]=\delta_{jk}\left(1+\beta\sum_{l\neq j}(M_{jl}+\overline{M}_{jl})+2\beta_1M_j\right)-(1-\delta_{jk})\beta(M_{jk}-\overline{M}_{jk}). \qquad (13)$$

Finally we introduce another type of B_N-type Dunkl operator:

$$D_j^{\pm} = D_j \mp \omega x_j \tag{14}$$

which satisfies similar relations between the D_j's and x_j's:

$$M_j D_j^{\pm} = -D_j^{\pm} M_j \qquad M_{jk} D_j^{\pm} = D_k^{\pm} M_{jk} \tag{15}$$

$$[D_j^{\pm}, D_k^{\pm}] = 0 \tag{16}$$

$$[D_j^{+}, D_k^{-}] = 2\omega[D_j, x_k]. \tag{17}$$

In fact, if we redefine $D_j^{\pm}$ as $D_j^{\pm}/\sqrt{2\omega}$, then $\{D_j, x_j\}$ and $\{D_j^{+}, D_j^{-}\}$ have the same algebraic structure.

We remark that similar results can be obtained by using the gauge transformed versions of the operators D_j and $D_j^{\pm}$:

$$\widehat{D}_j = \Delta(x)^{-1} D_j \Delta(x) = D_j + \beta \sum_{k \neq j} \left[\frac{1}{x_j - x_k} + \frac{1}{x_j + x_k} \right] + \beta_1 \frac{1}{x_j} \tag{18}$$

$$\widehat{D}_j^{\pm} = \widetilde{\Delta}(x)^{-1} D_j^{\pm} \widetilde{\Delta}(x) = \widehat{D}_j - (\omega \pm \omega) x_j \tag{19}$$

where $\Delta(x) = \prod_{j<k} (x_j^2 - x_k^2)^{\beta} \prod_l x_l^{\beta_1}$ and $\widetilde{\Delta}(x) = \Delta(x) \exp(-\frac{1}{2}\omega \sum_j x_j^2)$.

As in the ordinary case [7, 17], the integrals of motion for the B_N-type (spin) Calogero–Moser model and the B_N-type (spin) Calogero model can be constructed by using the Dunkl operators D_j and $D_j^{\pm}$, respectively. Moreover, under an appropriate transformation of the coordinates, the integrals of motion for the BC_N-type (spin) Calogero–Sutherland model are related to the operators $x_j D_j$. Then we shall unify the construction of these integrals of motion following [36]†. For this purpose, we introduce the operators

$$\Xi_j = (pD_j + qx_j)(p'D_j + q'x_j) \tag{20}$$

where $p, p', q, q' \in \mathbb{C}$. They satisfy the relations

$$M_j \Xi_j = \Xi_j M_j \qquad M_{jk} \Xi_j = \Xi_k M_{jk} \tag{21}$$

$$[\Xi_j, \Xi_k] = (pq' - p'q)\beta(\Xi_j - \Xi_k)(M_{jk} + \overline{M}_{jk}). \tag{22}$$

From the above formulae we can show the key formula

$$[\Xi_j^n, \Xi_k^m] = (pq' - p'q)\beta \sum_{a=1}^{m} \Xi_k^{m-a} (\Xi_j^n - \Xi_k^n) \Xi_j^{a-1} (M_{jk} + \overline{M}_{jk}). \tag{23}$$

Let us consider the quantities

$$\Upsilon_n = \sum_{j=1}^{N} \Xi_j^n. \tag{24}$$

Then the involutiveness of the Υ_n's is clear if $pq' - p'q = 0$. On the other hand, in general, using equation (23) and then explicitly antisymmetrizing in the index, we can prove the involutiveness of Υ_n's, i.e. $[\Upsilon_n, \Upsilon_m] = 0$. Moreover, from the B_N-symmetry of Υ_n, i.e. $[M_{jk}, \Upsilon_n] = [M_j, \Upsilon_n] = 0$, the projections $\pi(\Upsilon_n)$ are also involutive.

† Precisely speaking, this treatment is not convenient for the case of the B_N-type (spin) Calogero–Moser model, because the involutiveness of integrals is clear from their definition.

Specializing the parameters p, p', q and q', we define the two sets of the involutive operators $\{I_n^{\mathrm{CM}}\}_{n=1}^N$ and $\{I_n^{\mathrm{C}}\}_{n=1}^N$ corresponding to the B_N-type spin Calogero–Moser model and the B_N-type spin Calogero model, respectively:

$$I_n^{\mathrm{CM}} = \Upsilon_n \Big|_{\substack{p=p'=1\\ q=q'=0}} = \sum_{j=1}^{N} (D_j)^{2n} \tag{25}$$

$$I_n^{\mathrm{C}} = \Upsilon_n \Big|_{\substack{p=p'=1\\ -q=q'=\omega}} = \sum_{j=1}^{N} (D_j^+ D_j^-)^n. \tag{26}$$

Note that, in contrast to the ordinary (spin) Calogero–Moser model, the integrals I_n^{CM} depend only on D_j^2. This fact reflects the absence of translational invariance in the Hamiltonian (2). Note also that the $I_n^{CS} = \Upsilon_n \Big|_{\substack{p=0,\,p'=1\\ q=-1,\,q'=0}}$ are related to the BC_N-type spin Calogero–Sutherland model.

The Hamiltonian $\bar{H}_{\mathrm{C}}$ ($\bar{H}_{\mathrm{CM}}$) is expressed by the operator I_1^{CM} (I_1^{C}):

$$\bar{H}_{\mathrm{CM}} = -I_1^{\mathrm{CM}} \tag{27}$$

$$\bar{H}_{\mathrm{C}} = -I_1^{C} + \mathcal{E}_N^{(0)} \tag{28}$$

where $\mathcal{E}_N^{(0)} = \omega[N + 2\beta\sum_{j<k}(M_{jk} + \overline{M}_{jk}) + 2\beta_1 \sum_j M_j]$. It is thus clear that the I_n^{CM}'s are integrals of motion for the B_N-type spin Calogero–Moser model. It remains for us to show that the I_n^{C} 's commute with $\bar{H}_{\mathrm{C}}$. This can be checked by using the formula

$$[\bar{H}_{\mathrm{C}}]D_j^{\pm}] = \pm 2\omega D_j^{\pm}. \tag{29}$$

Hence the B_N-type spin Calogero–Moser model and the B_N-type spin Calogero model are integrable. As mentioned, using the projection π, we can obtain the corresponding integrals of motion which depend on the spin variables.

Let us now turn to the spin chain model related to the B_N-type spin Calogero model. We apply the standard technique due to Polychronakos [17] (see also [37, 38]). That is, we consider the strong coupling limit $\beta \to \infty$ in the Hamiltonian (3). Since the repulsion between particles and also between particles and mirror-image particles become dominant in the strong coupling limit, particles are enforced to localize with the positions x_j which are taken to minimize the potential

$$V(x) = \beta^2\tilde{\omega}^2 \sum_{j=1}^{N} x_j^2 + \beta^2 \sum_{1\leqslant j\neq k\leqslant N} \left[\frac{1}{(x_j - x_k)^2} + \frac{1}{(x_j + x_k)^2}\right] + \beta^2\gamma^2 \sum_{j=1}^{N} \frac{1}{x_j^2}. \tag{30}$$

Here we rescaled the coupling constant ω of the harmonic potential in order for the system to have a non-trivial limit. Also we rescaled $\beta_1 = \beta\gamma$. Note that $\tilde{\omega}$ can be absorbed into the definition of the x_j's. Then we set $\tilde{\omega} = 1$ hereafter. From $\partial_j V(x) = 0$, we can obtain the result that such x_j's satisfy the condition

$$2\sum_{k\neq j} \left[\frac{1}{(x_j - x_k)^3} + \frac{1}{(x_j + x_k)^3}\right] + \gamma^2 \frac{1}{x_j^3} = x_j. \tag{31}$$

The above formula is equivalent to the condition that the $y_j = x_j^2$ be zeros of the Laguerre polynomial $L_N^{(|\gamma|-1)}(y)$ [16].

In the strong-coupling limit $\beta \to \infty$, the elastic modes decouple from the internal degrees of freedom (the latter constitute the desired spin chain model):

$$\bar{H}_{\mathrm{C}} \longrightarrow H_{\mathrm{ela}} - \beta\bar{\mathcal{H}}_{\mathrm{PF}}. \tag{32}$$

Here H_{ela} represents the Hamiltonian for the elastic degrees of freedom and $\bar{\mathcal{H}}_{\mathrm{PF}}$ is the Hamiltonian which is obtained by respectively replacing P_{jk} and P_j with M_{jk} and M_j in (1), i.e. $\mathcal{H}_{\mathrm{PF}} = \pi(\bar{\mathcal{H}}_{\mathrm{PF}})$.

Let us define the operators

$$\mathcal{D}_j^{\pm} = \mathcal{D}_j \pm x_j \tag{33}$$

$$\Xi_j = \mathcal{D}_j^{+}\mathcal{D}_j^{-} = \mathcal{D}_j^2 - x_j^2 - \sum_{k\neq j}(M_{jk} + \overline{M}_{jk}) - \gamma M_j. \tag{34}$$

The operators $\mathcal{D}_j^{\pm}$ can be thought of as the large-β limit of the operators $D_j^{\pm}$. Thus we expect that the operators $\mathcal{I}_n^{\mathrm{PF}} = \sum_{j=1}^{N} \Xi_j^n$ are the integrals of motion for the B_N-PF model. We can show the involutiveness of the operators $\mathcal{I}_n^{\mathrm{PF}}$ along the same argument as those for the B_N-type spin Calogero model. The remaining task is to show the commutativity of $\mathcal{I}_n^{\mathrm{PF}}$ with $\bar{\mathcal{H}}_{\mathrm{PF}}$. Clearly, it suffices to show $[\bar{\mathcal{H}}_{\mathrm{PF}}, \Xi_j] = 0$. This can be proved as follows. It is easy to show that the following formula holds:

$$[\bar{H}_{\mathrm{CM}}, D_j] = 0 \Longleftrightarrow \left[-\sum_l \partial_l{}^2 - \beta\bar{\mathcal{H}}_{\mathrm{PF}} + \beta^2\mathcal{O}, \partial_j - \beta\mathcal{D}_j\right] = 0 \tag{35}$$

where

$$\mathcal{O} = \sum_{1\leqslant j\neq k\leqslant N}\left[\frac{1}{(x_j - x_k)^2} + \frac{1}{(x_j + x_k)^2}\right] + \gamma^2\sum_{j=1}^{N}\frac{1}{x_j^2}. \tag{36}$$

Let us consider the expansion of the relation (35) in powers of β. Since this relation holds for all β, each term must separately vanish. Thus the term of order β^2 gives

$$[\bar{\mathcal{H}}_{\mathrm{PF}}, \mathcal{D}_j] = [\partial_j, \mathcal{O}] = -4\sum_{k\neq j}\left[\frac{1}{(x_j - x_k)^3} + \frac{1}{(x_j + x_k)^3}\right] - 2\gamma^2\frac{1}{x_j^3}. \tag{37}$$

Also direct calculation shows that

$$[\bar{\mathcal{H}}_{\mathrm{PF}}, x_j] = -2\mathcal{D}_j. \tag{38}$$

Using the above two formulae (37), (38) and the properties $[\bar{\mathcal{H}}_{\mathrm{PF}}, M_{jk}] = [\bar{\mathcal{H}}_{\mathrm{PF}}, M_j] = 0$, we obtain

$$[\bar{\mathcal{H}}_{\mathrm{PF}}, \Xi_j] = ([\bar{\mathcal{H}}_{\mathrm{PF}}, \mathcal{D}_j] + 2x_j)\mathcal{D}_j + \mathcal{D}_j([\bar{\mathcal{H}}_{\mathrm{PF}}, \mathcal{D}_j] + 2x_j). \tag{39}$$

If the x_j's are chosen to take values in the set of square roots of the zeros of the Laguerre polynomial $L_N^{(|\gamma|-1)}(y)$, then we have $[\bar{\mathcal{H}}_{\mathrm{PF}}, \mathcal{D}_j] + 2x_j = 0$ ($\Leftrightarrow$ (31)), hence $[\bar{\mathcal{H}}_{\mathrm{PF}}, \Xi_j] = 0$.

Therefore we have proved the integrability of the B_N-PF model and have obtained the set of the integrals of motion $\{\pi(\mathcal{I}_n^{\mathrm{PF}})\}_{n=1}^{N}$ for this model. For example

$$\pi(\mathcal{I}_1^{\mathrm{PF}}) = -E_N - \left[\sum_{1\leqslant j\neq k\leqslant N}(P_{jk} + \overline{P}_{jk}) + 2\gamma\sum_{j=1}^{N}P_j\right] \tag{40}$$

where $E_N \in \mathbb{R}_{>0}$ is given by

$$E_N = \sum_{j=1}^{N}x_j^2 + \sum_{1\leqslant j\neq k\leqslant N}\left[\frac{1}{(x_j - x_k)^2} + \frac{1}{(x_j + x_k)^2}\right] + \gamma^2\sum_{j=1}^{N}\frac{1}{x_j^2}. \tag{41}$$

Finally, we would like to make some comments on algebraic interpretations of the results presented here. Our construction naturally leads to algebras of integrals of motion. For example, the Virasoro-like structure is given by

$$[J_n, J_m] = 0 \tag{42}$$

$$[L_n, J_m] = -mJ_{n+m} \tag{43}$$

$$[L_n, L_m] = (n-m)L_{n+m} \tag{44}$$

where

$$J_n = I_n^{\mathrm{CM}} \qquad (I_n^{\mathrm{C}}/(2\omega)^n) \tag{45}$$

$$L_n = \frac{1}{2}\sum_{j=1}^{N} x_j D_j^{2n+1} \qquad \left(\frac{1}{2}\sum_{j=1}^{N} D_j^-(D_j^+)^{2n+1}/(2\omega)^{n+1}\right). \tag{46}$$

Note that in (46) the total degree of the operator is always even as the polynomial of x_j and D_j (D_j^- and D_j^+). This fact reflects the absence of translational invariance. We can also construct the algebra of the integrals of motion related to the W_∞ algebra.

Other important features of the B_N-type spin Calogero model and the B_N-PF model are the existence of the spectrum generating algebras and the twisted Yangian symmetries. One of the authors has shown that the spectra of the B_N-type spin Calogero model are equally spaced [32]. It is easy to see that the same is true for the B_N-PF model. This is caused by the existence of the spectrum generating algebras (29) and

$$[\bar{\mathcal{H}}_{\mathrm{PF}}, \mathcal{D}_j^{\pm}] = \mp 2\mathcal{D}_j^{\pm}. \tag{47}$$

Moreover numerical studies show that the B_N-PF model possesses the 'super-multiplet' structure. The algebra underlying this structure is the twisted Yangian (see, for example, [39]). As in the ordinary cases [40, 41, 23, 36], the twisted Yangian symmetries of the B_N-type spin Calogero model and the B_N-PF model are easily seen from the transfer matrices of these systems which can be constructed by the Dunkl operators $D_j^{\pm}$ and $\mathcal{D}_j^{\pm}$.

Acknowledgments

We would like to thank H Awata, N Kawakami, Y Matsuo, S Odake and S-K Yang for discussions. TY was supported by the COE (Centre of Excellence) researchers program of the Ministry of Education, Science and Culture, Japan.

References

[1] Calogero F 1969 *J. Math. Phys.* **10** 2197
Sutherland B 1970 *J. Math. Phys.* **12** 246, 251; 1971 *Phys. Rev.* A **4** 2019; 1972 *Phys. Rev.* A **5** 1372

[2] Haldane F D M 1988 *Phys. Rev. Lett.* **60** 635
Shastry B S 1988 *Phys. Rev. Lett.* **60** 639

[3] For reviews, see the following papers and references therein:
Kawakami N 1994 *Prog. Theor. Phys.* **91** 189
Haldane F D M 1994 *Correlation Effects in Low Dimensional Electron Systems* ed A Okiji and N Kawakami (Berlin: Springer)

[4] Haldane F D M 1991 *Phys. Rev. Lett.* **66** 1529

[5] Bernard D and Wu Y-S 1995 *New Developments of Integrable Systems and Long-Ranged Interaction Models* ed M-L Ge and Y-S Wu (Singapore: World Scientific)

[6] Polychronakos A P 1992 *Phys. Rev. Lett.* **69** 703

[7] Minahan J A and Polychronakos A P 1993 *Phys. Lett.* **302B** 265
[8] Fowler M and Minahan J A 1993 *Phys. Rev. Lett.* **70** 2325
[9] Brink L, Hansson T H and Vasiliev M A 1992 *Phys. Lett.* **286B** 109
[10] Brink L and Vasiliev M A 1993 *Mod. Phys. Lett.* **8A** 3583
[11] Dodlov O V, Konstein S E and Vasiliev M A 1993 *JETP Lett.* **58** 855
[12] Wang D F and Gruber C 1994 *Phys. Rev.* B **49** 16712
[13] Ujino H and Wadati M 1995 *J. Phys. Soc. Japan* **64** 4121
[14] Calogero F 1977 *Lett. Nuovo Cimento* **19** 505; **20** 251; 489; 1983 *Nonlinear Phenomena (Lecture Notes in Physics 189)* (Berlin: Springer)
[15] Perelomov A M 1978 *Ann. Inst. H. Poincaré* **28** 407
[16] Ahmed S, Brushi M, Calogero F, Olshanetsky M A and Perelomov A M 1979 *Nuovo Cimento* B **49** 173
[17] Polychronakos A P 1993 *Phys. Rev. Lett.* **70** 2329; 1994 *Nucl. Phys.* B **419** 553
[18] Ha Z N C and Haldane F D M 1992 *Phys. Rev.* B **46** 9359
[19] Hikami K and Wadati M 1993 *J. Phys. Soc. Japan* **62** 4203
[20] Hikami K and Wadati M 1993 *J. Phys. Soc. Japan* **62** 469; *Phys. Lett.* **173A** 263
[21] Vacek K, Okiji A and Kawakami N 1994 *J. Phys. A: Math. Gen.* **27** L201; *Phys. Rev.* B **49** 4635
[22] Frahm H 1993 *J. Phys. A: Math. Gen.* **26** L437
[23] Hikami K 1995 *Nucl. Phys.* B **441** 530
[24] Simons B D and Altshuler B L 1994 *Phys. Rev.* B **50** 1102
[25] Bernard D, Pasquier V and Serban D 1995 *Europhys. Lett.* **30** 301
[26] Olshanetsky M A and Perelomov A M 1981 *Phys. Rep.* **71** 313; 1983 *Phys. Rep.* **94** 313
[27] Cherednik I 1994 *Adv. Math.* **106** 65
[28] Kapustin A and Skorik S 1994 *Phys. Lett.* **196A** 47
[29] Beenakker C W J and Rejaei B 1994 *Phys. Rev.* B **49** 7499
[30] Caselle M 1995 *Phys. Rev. Lett.* **74** 2776
[31] Yamamoto T, Kawakami N and Yang S-K 1996 *J. Phys. A: Math. Gen.* **29** 317
[32] Yamamoto T 1995 *Phys. Lett.* **208A** 293
[33] Szegö G 1939 *Orthogonal Polynomials (American Mathematical Society Colloquium Publishing 23)* (Providence, RI: American Mathematical Society)
[34] Dunkl C F 1989 *Trans. Am. Math. Soc.* **311** 167
[35] Kuznetsov V B 1995 Hidden symmetry of the quantum Calogero–Moser system *Preprint* solv-int/9509001
[36] Bernard D, Hikami K and Wadati M 1995 *New Developments of Integrable Systems and Long-Ranged Interaction Models* ed M-L Ge and Y-S Wu (Singapore: World Scientific)
[37] Sutherland B and Shastry B S 1993 *Phys. Rev. Lett.* **71** 5
[38] Kato Y and Kuramoto Y 1995 *Phys. Rev. Lett.* **74** 1222
[39] Molev A, Nazarov M and Olshanskii G 1994 Yanginas and Classical Lie Algebras *Preprint* hep-th/9409025
[40] Haldane F D M, Ha Z N C, Talstra J C, Bernard D and Pasquier V 1992 *Phys. Rev. Lett.* **69** 2021
[41] Bernard D, Gaudin M, Haldane F D M and Pasquier V 1993 *J. Phys. A: Math. Gen.* **26** 5219